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NATURAL CONVECTION FLOWS ADJACENT TO HORIZONTAL SURFACES RESULTING FROM THE COMBINED BUOYANCY EFFECTS OF THERMAL AND MASS DIFFUSION

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Abstract—Consideration is given to the natural convection flow over a horizontal, isothermal and semi-infinite surface, generated by the combined buoyancy effects of surface heating and diffusion of chemical species.

The governing differential equations are developed in terms of a similarity variable which allows for the effect of mass diffusion. Some numerical solutions to the differential equations are presented.

NOMENCLATURE

- C_p, specific heat of the fluid at constant pressure;
- D, chemical molecular diffusivity;
- f, nondimensional stream function;
- g, gravitational acceleration;
- Gr, Grashof number defined by equation (9);
- k, thermal conductivity of the fluid;
- Nu, Nusselt number;
- Pr, Prandtl number $= \nu/\alpha$;
- $\dot{Q}$, convection rate of thermal energy;
- Sc, Schmidt number $= \nu/D$;
- x, horizontal distance along the surface;
- y, distance away from the surface;
- α, thermal molecular diffusivity;
- β, volumetric coefficient of thermal expansion;
- β^*, volumetric coefficient of expansion with concentration;
- ϕ, nondimensional temperature;
- μ, dynamic viscosity of the fluid;
- ν, kinematic viscosity of the fluid;
- ψ, stream function;
- η, similarity variable;
- ρ, fluid density.

Subscripts

- C, based on species concentration level;
- x, based on distance x;
- t, based on temperature level;
- 0, at the surface;
- ∞, in the undisturbed fluid.

INTRODUCTION

SOME natural convection flows in the atmosphere and micro-meteorological phenomena are often not caused entirely by the effect of temperature gradients but also by the differences in concentrations of dissimilar chemical species. The heating of the earth by sunlight causes atmospheric thermal convection, which may be modified by the presence of moisture evaporated from the ground. Many other important problems of present and future interest concern the diffusion of thermal and chemical-species buoyant streams discharged into bodies of water. These convection and transfer processes are governed by buoyancy mechanisms arising from both thermal and species diffusion. The second is often the largest effect. In a large number of important technological applications the flow is also simultaneously driven by the joint action of temperature and chemical composition differences or gradients. Thus, knowledge concerning the combined effect of heat and mass diffusion is important for future technological and environmental studies.

Vertical natural convection flows resulting from these combined buoyancy mechanisms has been studied in the past. Gebhart and Pera [1] reviewed past experimental studies and analyses and presented a general formulation that accounts for combined driving mechanisms together with numerical results covering most of the practical applications. This formulation dispenses with most approximations and applies to a wide range of important flows. The same formulation will also be used in the present study.

The present analysis treats the flow induced by the combined buoyancy effects due to thermal and chemical species diffusion adjacent to horizontal surfaces having uniform surface conditions with the buoyancy effect primarily away from the surface. The numerical calculations are restricted to air and to the diffusing molecular species of greatest practical importance. The equations and boundary conditions used are limited to processes which occur at low concentration differences since the boundary conditions at the surface are assumed unaffected by interfacial velocities. Species thermal diffusion and species diffusion of thermal energy, sometimes important in gases, are also neglected in the analysis.

ANALYSIS

The boundary-layer equations that govern the natural convection flow caused by the combined buoyancy effects of thermal and chemical species concentration differences are written below in terms of fluid velocity (u, v), dynamic pressure, (p), temperature (t) and concentration of a single diffusing chemical species (C). The equations apply for flows with a net buoyancy force away from the surface. They are simplified by the Boussinesq approximation and by the assumption that the concentration level of the diffusing species is small compared with other concentrations.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} + g(\beta\Delta t + \beta^*\Delta C) \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

$$u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y} = \alpha\frac{\partial^2 t}{\partial y^2} \tag{4}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial y^2} \tag{5}$$

The combined thermal and mass concentration buoyancy force term only appears in the momentum balance normal to the surface (y), the buoyancy force in the tangential direction (x) being always zero. The driving mechanism only acts indirectly through the dynamic pressure term in the x-momentum equation.

A similarity formulation for the above system of differential equations without a mass diffusion effect was previously shown to exist and some numerical solutions of the differential equations were reported by Rotem and Claassen [2] for the condition of uniform surface temperature. Pera and Gebhart [3] extended the analysis to include uniform temperature and uniform flux conditions and flows over slightly inclined surfaces. Experimental results were also presented to determine the limits of validity of the similarity solution.

For the complete set of the coupled differential equations, accounting for the mass diffusion, a similarity analysis is also possible for certain surface temperature and surface mass concentration variations. By neglecting temperature and species stratification in the distant medium and considering surface variations of the form:

$$t_0 - t_\infty = N_t x^n, \quad C_0 - C_\infty = N_C x^m, \tag{6}$$

a simple similarity variable exists when $n = m$. The condition $n = 0$ accounts for uniform surface temperature and concentration conditions and $n = \frac{1}{3}$ results for uniform flux surface conditions. N_t and N_C are appropriate constants. Any value of n is algebraically permissible but limitations of physical realism with respect to boundary layer thickness at the leading edge and conditions on the local flow require $-0{\cdot}5 \leqslant n \leqslant 2$. Following the analysis of Gebhart and Pera [1], the combined thermal and chemical species buoyancy effects are reduced to similarity form in the variable η:

$$\eta = \frac{y}{x}\sqrt[5]{\left(\frac{PGr_{x,t} + QGr_{x,C}}{5}\right)} = \frac{y}{x}\sqrt[5]{\left(\frac{Gr_{x,t}}{5}\right)}\sqrt[5]{(P + QN)} \tag{7}$$

where

$$N = \frac{Gr_{x,C}}{Gr_{x,t}} = \frac{\beta^*(C_0 - C_\infty)}{\beta(t_0 - t_\infty)} \tag{8}$$

is a parameter that determines the relative importance of chemical species and thermal diffusion in causing buoyancy and P and Q are convenient scaling factors that retain the combination $(P + QN)$ greater than zero for all practical conditions. The two local Grashof numbers are as usual defined as:

$$Gr_{x,C} = \frac{g\beta^* x^3(C_0 - C_\infty)}{\nu^2}$$

and (9)

$$Gr_{x,t} = \frac{g\beta x^3(t_0 - t_\infty)}{\nu^2}.$$

The dependent variables of temperature, concentration, velocity and pressure are generalized in terms of the surface temperature and concentration excess over far field conditions, and in terms of the stream and pressure functions, as follows:

$$\phi(\eta) = \frac{t - t_\infty}{t_0 - t_\infty}, \quad C(\eta) = \frac{C - C_\infty}{C_0 - C_\infty}, \tag{10}$$

$$\psi(x,y) = 5\nu\sqrt[5]{\left(\frac{Gr_{x,t}}{5}\right)}\sqrt[5]{(P + QN)}\, f(\eta), \tag{11}$$

$$p(x,y) = \frac{5\nu^2\rho}{x^2}\sqrt[5]{\left[\left(\frac{Gr_{x,t}}{5}\right)^4\right]}\sqrt[5]{[(P + QN)^4]}\, G(\eta) \tag{12}$$

In this formulation the parameter N may vary over a wide range. N is zero for no mass diffusion, infinite for no thermal diffusion, can be positive when both buoyancy effects are aiding the flow and negative when they oppose each other.

Equations (1)–(5) when, rewritten in similarity form, are:

$$f''' + (3 + n)ff'' - (2n + 1)f'^2 = \frac{\left(\frac{4n + 2}{5}\right)G + \left(\frac{n - 2}{5}\right)G'\eta}{(P + QN)} \tag{13}$$

$$G' = \phi + NC \tag{14}$$

$$\phi'' + Pr[(3 + n)f\phi' - 5n\phi] = 0 \tag{15}$$

$$C'' + Sc[(3 + n)fC' - 5nC] = 0 \tag{16}$$

where the primes indicate differentiation with respect to the similarity variable η.

Boundary conditions for the above set of equations are simple if we neglect any induced velocity at the surface caused by the mass diffusion effect. When very low species concentration is involved the boundary conditions are:

$$f(0) = f'(0) = G(0) = 1 - \phi(0) = 1 - C(0) = 0$$
$$f'(\infty) = \phi(\infty) = C(\infty) = 0 \tag{17}$$

Similar to the problem of the flow adjacent to a vertical surface, (see Gebhart and Pera [1]) very simple results may be found when the fluid Prandtl and Schmidt numbers are equal. In that case, the temperature and concentration distributions functions are identical

$[\phi(\eta) = C(\eta)]$, and only one of the equations (15) and (16) need to be integrated across the boundary layer, with the appropriate boundary conditions. The problem thus reduces to that for a single buoyancy mechanism and these same results may be used.

Heat and mass transport quantities may be computed from the distributions of the temperature and mass concentration functions at the surface:

$$\frac{Nu_{x,t}}{\sqrt[5]{(Gr_{x,t})}} = -\frac{\phi'(0)}{\sqrt[5]{5}}\sqrt[5]{(P + QN)} \qquad (18)$$

and

$$\frac{Nu_{x,C}}{\sqrt[5]{(|Gr_{x,C}|)}} = -\frac{C'(0)}{\sqrt[5]{5}}\sqrt[5]{\left(\left|\frac{P + QN}{N}\right|\right)} \qquad (19)$$

where $\phi'(0)$ and $C'(0)$ are functions of the Prandtl and Schmidt number and of the buoyancy parameter N and are generally the results of a numerical integration.

The amount of heat, of chemical species, of momentum and of mass convection rate in the boundary layer at any location (x) are usually important in any practical situation, therefore, these quantities are calculated below in terms of similarity variables. The heat convected is:

$$\dot{Q} = 5\nu\rho C_p N_t I_t \left\{\frac{g\beta N_t}{5\nu^2}\right\}^{\frac{1}{5}} x^{\frac{6n+3}{5}} (P + QN)^{\frac{1}{5}}. \qquad (20)$$

I_t is the numerical value of the following integral

$$I_t = \int_0^\infty \phi f' \, d\eta. \qquad (21)$$

The case when $\dot{Q}$ is not a function of x ($n = -\frac{1}{2}$) represent a situation in which all the heat is released from the leading edge of the plate, i.e. $\dot{Q}$ is independent of x. This is certainly the lower limit for n. Determination of equivalent expressions for the mass of the convected chemical species leads to the integral:

$$I_C = \int_0^\infty Cf' \, d\eta. \qquad (22)$$

Similarly the total mass and momentum flow at x may be obtained from integrals of the form:

$$J = \int_0^\infty f' \, d\eta \quad \text{and} \quad L = \int_0^\infty f'^2 \, d\eta. \qquad (23)$$

The eighth-order system of differential equations (13)–(16) were numerically solved with the appropriate boundary conditions at the wall and in the far field and the values of the integrals (21)–(23) were also evaluated. The numerical procedure employed here is similar to that used by Pera and Gebhart [3] to solve the simpler sixth-order system resulting for flows driven entirely by thermal diffusion caused buoyancy. The numerical integration was performed with a fourth-order Runge–Kutta scheme, in combination with a linear interpolator that provides corrections to the assumed values of unknown boundary conditions at the surface. The present problem presents greater numerical difficulty due to the increased number of boundary conditions that must be asymptotically satisfied as $\eta \to \infty$.

The velocity, pressure, temperature and concentration distributions, together with the heat and mass transfer results, were computed for $Pr = 0{\cdot}7$ and for a range of the Schmidt numbers (0·1–10). This covers most of the important gases and vapors that commonly diffuse into the air (Water vapor ($Sc = 0{\cdot}6$), hydrogen ($Sc = 0{\cdot}22$), carbon dioxide ($Sc = 0{\cdot}94$) and ethyl benzene ($Sc = 2{\cdot}01$).

RESULTS AND DISCUSSIONS

The heat transfer and flow conditions over a heated horizontal surface where gases or liquid vapors diffuse into the main fluid, are modified by the combined effect of the ordinary thermal natural convection and the additional buoyancy mechanism produced by concentration differences of the diffusing chemical species. In the present paper we present numerical solutions of the governing equations (13)–(16) for a varied combination of flow conditions for uniform surface conditions, i.e. $n = m = 0$. Parameters governing the transfer of heat and mass have

Table 1. Flow and transport quantities for flow adjacent to isothermal horizontal surfaces. Pr = 0·7

Sc	N	$f''(0)$	$-G(0)$	$-\phi'(0)$	$-C'(0)$	J	L	I_t	I_C
0·1	0·5	0·7226	2·4619	0·5852	0·2259	1·3550	0·3656	0·2780	0·7155
	1·0	0·8153	3·6890	0·6158	0·2400	1·5156	0·4627	0·2926	0·7742
	2·0	0·9007	6·1295	0·6408	0·2514	1·6353	0·5479	0·3044	0·8178
0·5	−0·4	0·4814	0·7441	0·4367	0·3488	−0·0684	0·1267	0·2040	0·2195
	0·5	0·5426	1·9705	0·5016	0·4260	0·6987	0·1422	0·2383	0·2824
	1·0	0·5545	2·6759	0·5070	0·4309	0·7125	0·1494	0·2412	0·2861
	2·0	0·5660	4·0843	0·5123	0·4357	0·7308	0·1567	0·2438	0·2895
0·7	All	0·5191	1·2683	0·4890	0·4890	0·6457	0·1262	0·2326	0·2326
1·0	−0·5	0·5783	0·6988	0·5148	0·6056	0·7098	0·1553	0·2494	0·2018
	0·5	0·4982	1·8376	0·4788	0·5656	0·6183	0·1154	0·2277	0·1884
	1·0	0·4875	2·4070	0·4732	0·5595	0·6025	0·1097	0·2250	0·1864
	2·0	0·4765	3·5454	0·4673	0·5530	0·5857	0·1038	0·2221	0·1842
5·0	−0·5	0·7115	0·8857	0·5524	1·2403	0·7449	0·1880	0·2629	0·0826
	0·5	0·4433	1·6373	0·4577	1·0437	0·5938	0·1006	0·2176	0·0695
	1·0	0·4019	2·0031	0·4380	1·0046	0·5583	0·0859	0·2080	0·0700
	2·0	0·3572	2·7328	0·4136	0·9579	0·5132	0·0694	0·1961	0·0640
10·0	−0·5	0·7377	0·9338	0·5570	1·6174	0·7466	0·1904	0·2650	0·5557
	0·5	0·4312	1·5821	0·4545	1·3380	0·5522	0·0993	0·2160	0·0445
	1·0	0·3823	1·8896	0·4323	1·2799	0·5557	0·0838	0·2052	0·0426
	2·0	0·3286	2·4996	0·4040	1·2084	0·5073	0·0661	0·1914	0·0402

also been computed. Table 1 summarizes the flow, pressure, heat and mass transfer parameters such as $f''(0)$, $G(0)$, $\phi'(0)$, and $C'(0)$, for $Pr = 0{\cdot}7$ for a range of Schmidt number of common interest. For $Pr = Sc$ equations (15) and (16) are identical and $\phi'(0) = C'(0)$ therefore these results are as those computed before (Rotem and Claassen [2] and Pera and Gebhart [3]). In Table 1 are also given numerical values of the integrals (21)–(23) for the above conditions. These may be used to determine the total amount of heat, of chemical species, of momentum and of total mass convected in the boundary layer at a given location (x).

Figures 1 and 2 show temperature, concentration, velocity and pressure distributions for $Pr = Sc$ and $N = -0{\cdot}5$, 0, 1 and 2, where P and Q are taken as 1·0. The advantages of using different values of the scaling parameters P and Q for extreme conditions of opposed driving mechanisms were given by Gebhart and Pera [1]. Since in the present work such extreme regions are not studied, the values of P and Q are taken as 1·0 throughout. It is seen that the value of N has no effect on the ϕ and C distribution in terms of the combined driving force $Gr_{x,t}$ and $Gr_{x,C}$ but affects the magnitude of the velocity and pressure functions. Figures 3–6 show the same distribution functions for various values of Schmidt numbers. For $Pr \neq Sc$, the thermal and species diffusion boundary regions thicknesses are of different extent. The aiding and opposing buoyancy effects alter the form of these distributions, depending on the magnitude of N. For $Sc > Pr$ and with a given negative value of N a flow reversal could appear in the inner layer close to the surface (not shown in the figures), a similar effect appears in the outer layer when $Sc < Pr$ as seen in Fig. 3. Under these conditions the boundary layer equations used in the analysis are in question since the component of the velocity normal to the surface (v) becomes of the order of the tangential component (u). For those situations, therefore, the approximations made must be reconsidered.

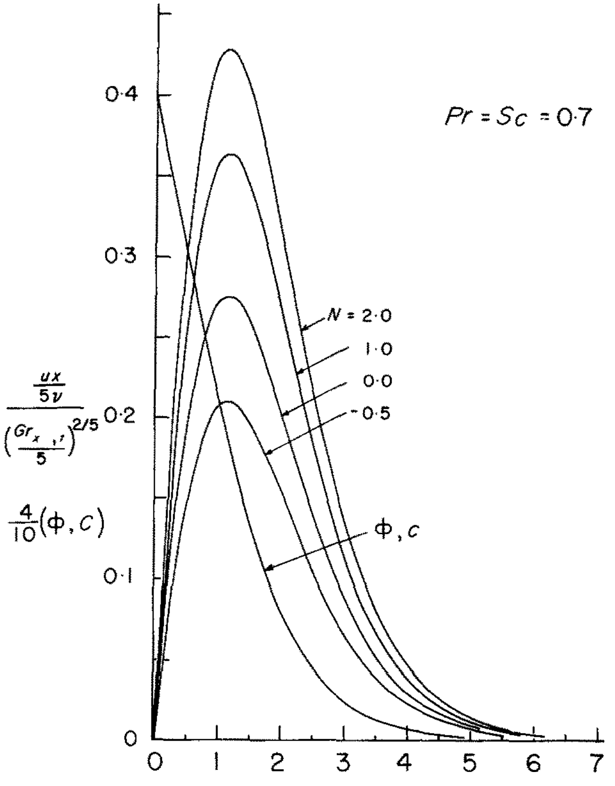

FIG. 1. Temperature, concentration and velocity distributions on the boundary region for various conditions ($N = -0{\cdot}5$, 0, 1 and 2) of combined buoyancy effects.

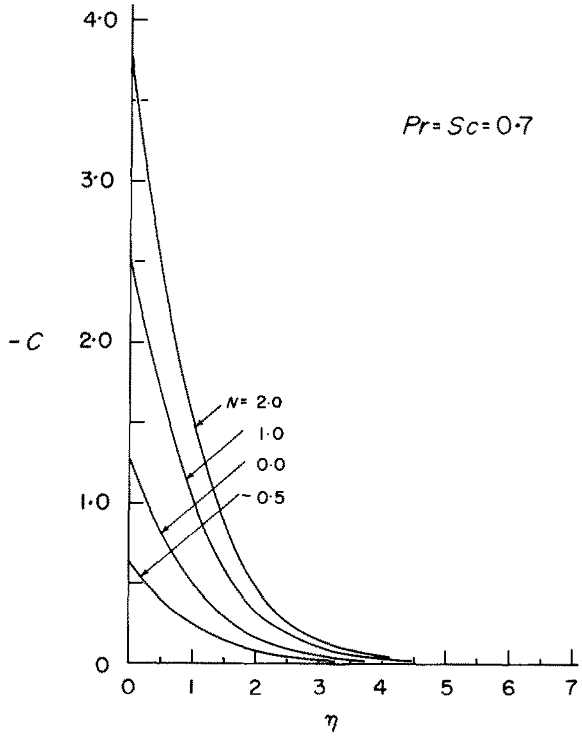

FIG. 2. Pressure distribution on the boundary region for various conditions ($N = -0{\cdot}5$, 0, 1 and 2) of combined buoyancy effects.

In Figs. 3–6 the results are plotted in terms of $Gr_{x,t}$ so that the physical effects of the mass diffusion over the flow driven simply by thermal diffusion may be seen. Figures 4 and 6 indicate the extent of the temperature and concentration boundary regions. For a given Prandtl number, the effect of increasing the Schmidt number is to thin the concentration boundary layer and thicken the corresponding temperature layer; for a given Prandtl and Schmidt number, an increase of the aiding mass diffusion effect results in a thinning of both layers. Similar conclusions are reached for the pressure and velocity distributions (Figs. 3 and 5). However, these distributions demonstrate a stronger effect of the coupling with the mass diffusion mechanism.

Figure 3 indicates a strong flow reversal in the outer boundary region for negative values of N for $Sc = 0{\cdot}5$. A curve for $N = -0{\cdot}4$ is given to show this effect.

Figures 7 and 8 present the surface heat and mass transfer parameters for $Pr = 0{\cdot}7$, $Sc = 0{\cdot}1$, 0·7, 1, 5 and 10 for a range of N in both the aiding and opposing regions. Figure 7 shows the effect that the additional mechanism has on the heat transfer. All the curves coincide at $N = 0$, i.e. for pure thermal convection. N positive increases the heat transfer and the increment depends strongly on the physical

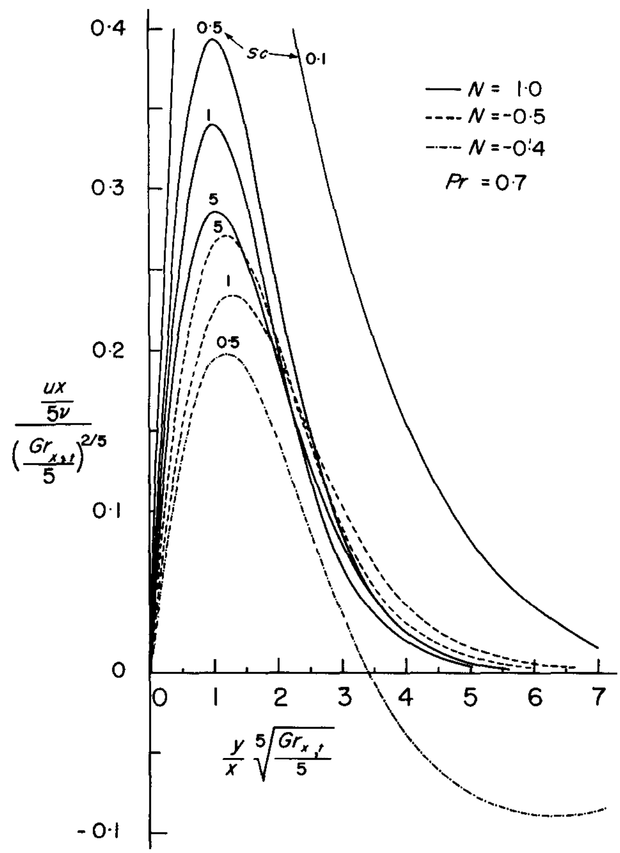

FIG. 3. Effect on flow velocity distribution of varying Schmidt numbers for the Prandtl number of air. $N = -0{\cdot}5$ and 1.

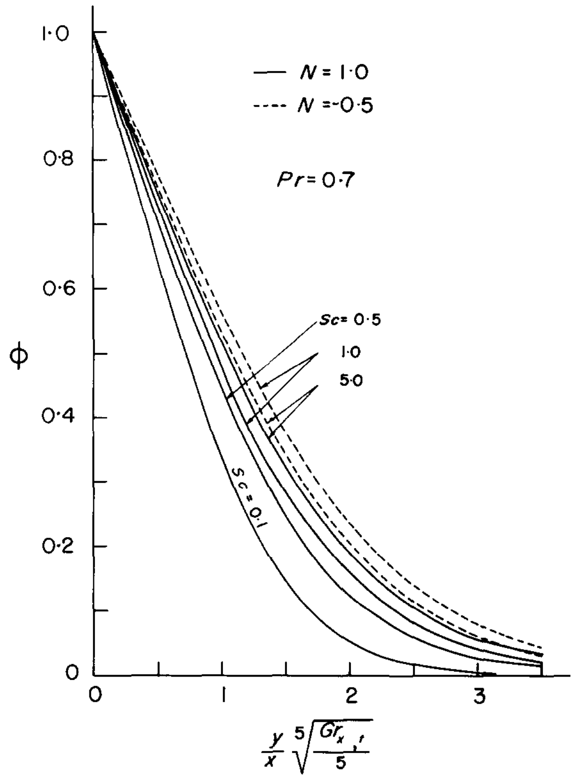

FIG. 4. Effect on temperature distribution of varying Schmidt numbers for the Prandtl number of air. $N = -0{\cdot}5$ and 1.

properties of the diffusing species as indicated by the Schmidt number. A negative value of N produces an opposite effect; note that with the present formulation ($P = Q = 1$), the lower possible limit for N is -1. For even stronger opposing mechanisms ($N < -1$), different values of P and Q may be chosen such as to maintain the relation $(P + QN) > 0$ in the region of interest.

Figure 8 shows the variation of the mass transfer parameter over the same range of N. Again $N = 0$ indicates the absence of mass diffusion and it is a singular point in the present formulation. Values of N near zero result in large values of the mass transport parameters and the species diffusion mechanism for very low concentration, therefore, is very efficient when expressed in terms of $Gr_{x,C}$. Under these conditions the flow is induced almost entirely by thermal buoyancy. As N increases the situation eventually reverses and asymptotic levels are approached in which the mass-induced buoyancy dominates. For $Pr = Sc$ the curve is asymptotic to the known result for purely thermal diffusion driven natural convection flow when $Pr = 0{\cdot}7$. For $Pr \neq Sc$ the curves go to the asymptotic values which pertain for each particular Schmidt number.

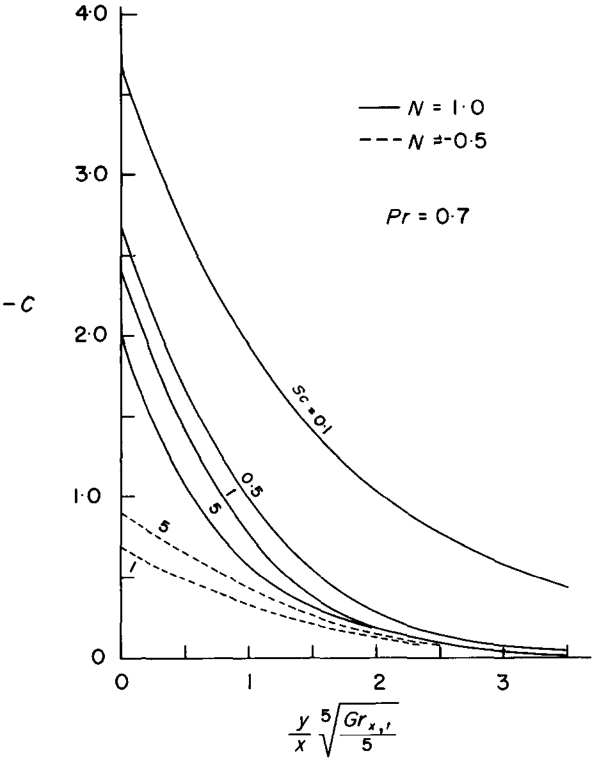

FIG. 5. Effect on motion distribution of varying Schmidt numbers for the Prandtl number of air. $N = -0{\cdot}1$ and 1.

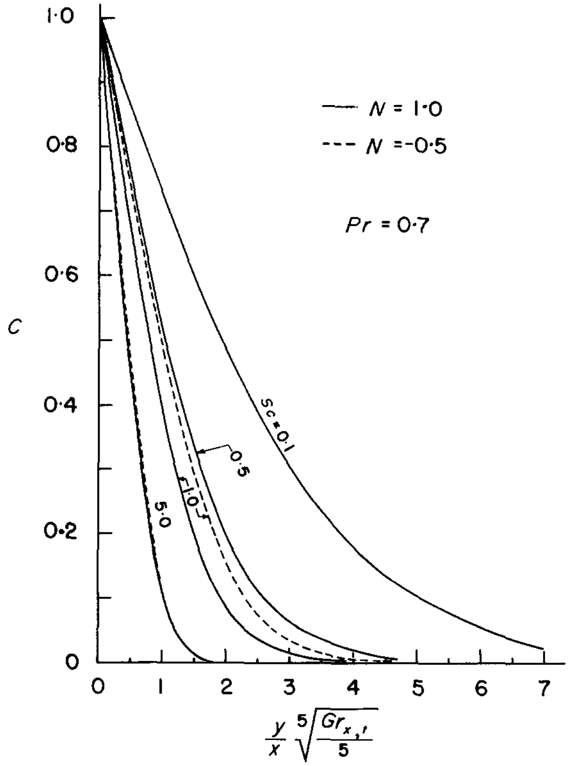

FIG. 6. Effect on species concentration distribution of varying Schmidt numbers for the Prandtl number of air. $N = -0{\cdot}1$ and 1.

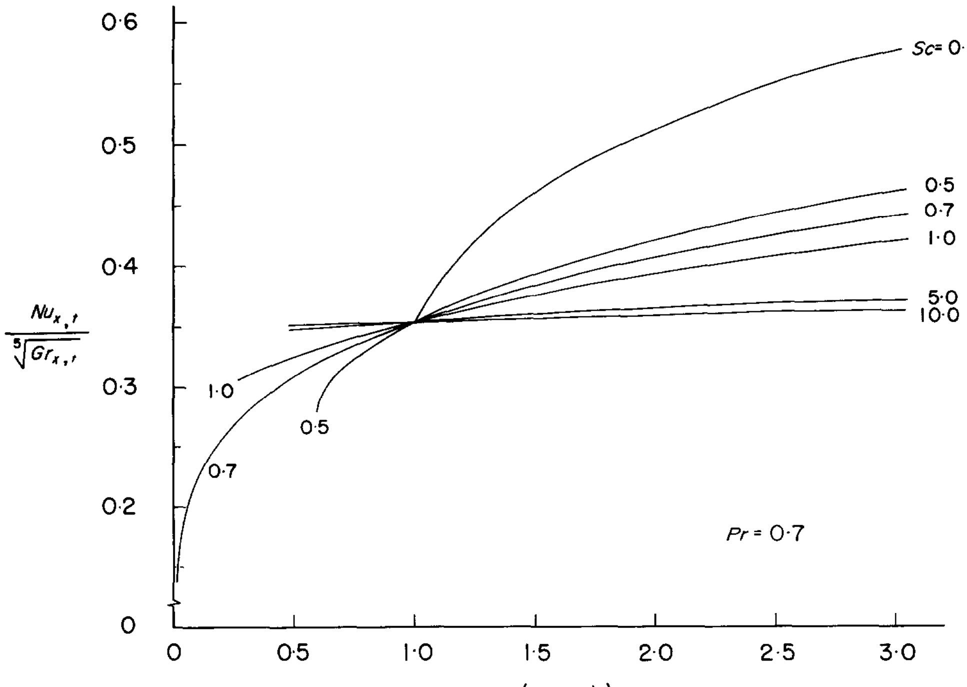

FIG. 7. Heat transfer parameter at various Schmidt numbers and for the Prandtl number of air, as a function of the relative value (N) of the two buoyancy mechanisms.

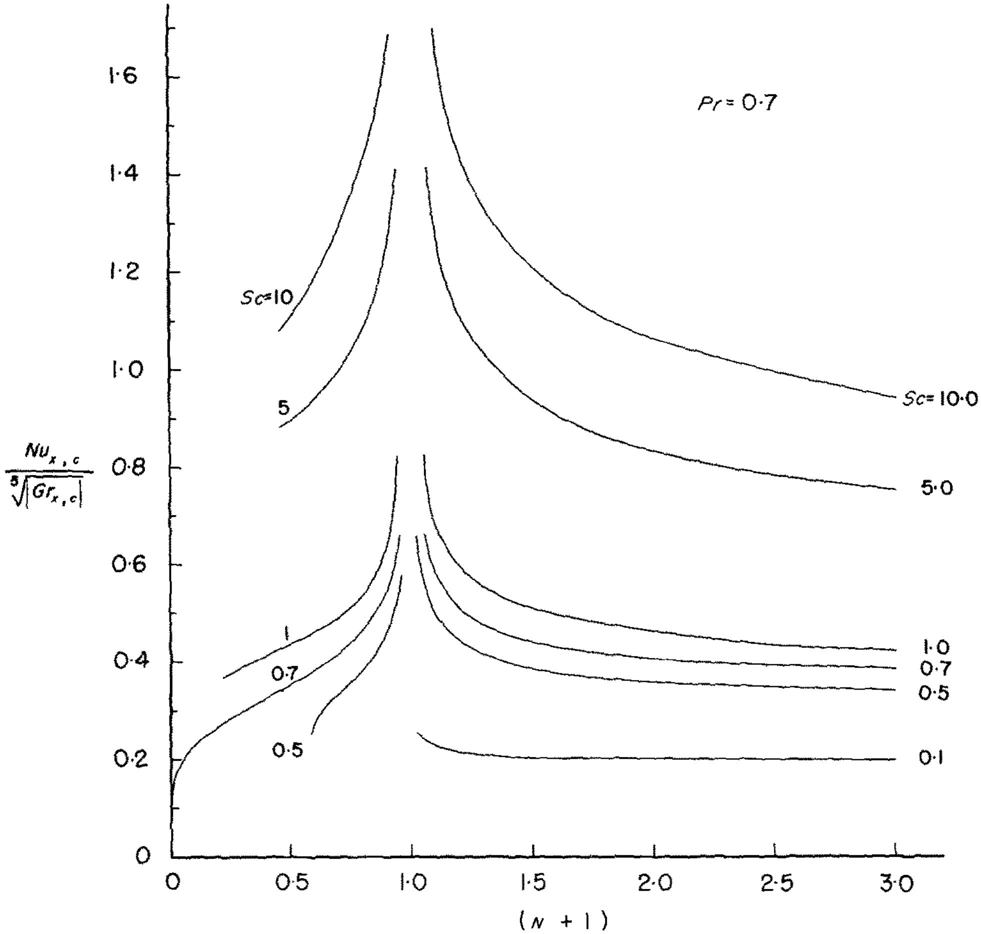

FIG. 8. Species transfer parameter at various Schmidt numbers and for the Prandtl number of air, as a function of the relative value (N) of the two buoyancy mechanisms.

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ECOULEMENTS DE CONVECTION NATURELLE ADJACENTS À DES SURFACES HORIZONTALES, SOUS LES EFFETS COMBINÉS DE LA DIFFUSION THERMIQUE ET MASSIQUE

Résumé—On considère l'écoulement de convection naturelle sur une surface horizontale, isotherme et semi-infinite, engendré par les effets combinés du chauffage superficiel et de la diffusion d'espèces chimiques. Les équations différentielles décrivant le phénomène sont développées en termes d'une variable de similarité qui rend compte de l'effet de la diffusion massique. On présente quelques solutions numériques des équations différentielles.

NATÜRLICHE KONVEKTION AN HORIZONTALEN OBERFLÄCHEN DURCH GLEICHZEITIGEN WÄRME- UND STOFFÜBERGANG.

Zusammenfassung—Es werden Überlegungen angestellt über die natürliche Konvektion über einer horizontalen, isothermen halbunendlichen Oberfläche, hervorgerufen durch kombinierte Auftriebseffekte infolge von Oberflächenheizung und Diffusion von chemischen Stoffen. Die massgebenden Differentialgleichungen werden aufgestellt als Funktion einer Ähnlichkeitsvariablen, die die Stoffdiffusion berücksichtigt. Einige numerische Lösungen der Differentialgleichungen werden angegeben.

СВОБОДНОКОНВЕКТИВНЫЕ ТЕЧЕНИЯ У ГОРИЗОНТАЛЬНЫХ ПОВЕРХНОСТЕЙ В РЕЗУЛЬТАТЕ ОДНОВРЕМЕННОГО ДЕЙСТВИЯ СИЛ, ОБУСЛОВЛЕННЫХ ДИФФУЗИЕЙ ТЕПЛА И МАССЫ

Аннотация—Рассмотрено свободноконвективное течение на горизонтальной и полубесконечной поверхности, возникающее в результате одновременного действия движущих сил, обусловленных нагревом поверхности и диффузией химических веществ. Выведены основные дифференциальные уравнения с помощью переменной подобия, которая учитывает влияние диффузии массы. Представлены некоторые численные решения дифференциальных уравнений.